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## Local improving algorithms for large cuts in graphs with maximum degree three

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### Abstract

We analyze ‘local switching’ search algorithms for finding large bipartite subgraphs in simple undirected graphs. The algorithms are based on the ‘measure of effectiveness’ of the partitions of the vertex set. We analyze the worst-case behaviour of these algorithms, giving general lower bounds. Using a vertex and its neighbours, we define the improving and indifferent switchings and, indexed by two numbers  $(m, n)$ , procedures to improve the reading cut. Since the concept of switching has its limits, we indicate how larger the substructures should be taken to improve locally optimal solution. © 2001 Elsevier Science B.V. All rights reserved.

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### 1. Introduction and preliminaries

Karp [4] proved, that the problem of finding a largest bipartite subgraph contained in a given graph is NP-complete. Yannakakis [11] has shown that the problem remains NP-complete, even if the graph considered is 3-regular and triangle free. See also Loebl [6] and Tovey [10] for discussion of complexity of local search algorithms. While finding a largest subgraph of this kind appears to be difficult, we are interested in polynomial algorithms which guarantee good lower bounds on the number of edges of bipartite subgraphs. The local improving algorithms are polynomial and also belong to the class of parallel algorithms. We refer to Poljak and Tuza [8] for a detailed literature review concerning large bipartite subgraphs. Bondy and Locke [1] present a local switching search algorithm which returns a bipartite subgraph containing at least  $\frac{4}{5}$  edges of triangle-free graphs with maximum degree 3. We extend their research on all graphs with maximum degree 3.

Let  $G = (V, E)$ , where  $V$  is a vertex set and  $E$  is an edge set, be an undirected graph without loops and multiple edges. We will denote by  $Bp(G)$  the set of all bipartitions of the vertex set  $V$ , i.e. the set of all pairs  $(A, B)$  such that  $A \subset V$  and  $B = V \setminus A$ . We will denote by  $[A, B]_G$  (or simply  $[A, B]$ ) the set of all edges having a vertex in  $A$  and

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the second one in  $B$ . The subgraph  $(V, [A, B])$  is called the *bipartite subgraph defined by  $(A, B)$* . A sequence of vertices  $(v_0, v_1, \dots, v_n)$ ,  $n > 0$ , is a *path of the length  $n$*  iff  $\{v_{i-1}, v_i\} \in E$  for each  $i = 1, \dots, n$  and every two edges are different. It is an *alternating path* iff it is also a path in the bipartite subgraph defined by  $(A, B)$ . For a given vertex  $v$ , by the *star* (or *first star*) of  $v$  we mean the set of all vertices-neighbours of  $v$ . The  *$n$  star* is the set of all vertices obtained from  $v$  by a path not longer than  $n$ .

We will denote by  $d(v, X)$  the number of all edges from  $v$  to  $X \subset V$ . The degree of a vertex  $v$  is equal  $d(v, V)$ . Here  $K_r$  denotes the complete graph on  $r$  vertices. Each subgraph, isomorphic to  $K_3$ , is a triangle and it is a *pendant triangle* if two of its vertices have degree 2 in  $G$ . In the sequel,  $\Gamma$  denotes *the family of all graphs with degree no greater than 3 and without components isomorphic to either  $K_3$  or  $K_4$* .

Let us define the function

$$\mu(v, A, B) := \begin{cases} d(v, B) - d(v, A) & \text{if } v \in A, \\ d(v, A) - d(v, B) & \text{if } v \in B, \end{cases}$$

- the *weight* of the vertex  $v$  with respect to the bipartition  $(A, B)$  – and the function  $\mu_G$ ,

$$\mu_G(A, B) := \sum_{v \in V} \mu(v, A, B),$$

- the measure of effectiveness of bipartitions. It is easy to check, see also [2], that

$$|[A, B]_G| = \frac{1}{2}|E(G)| + \frac{1}{4}\mu_G(A, B). \quad (1)$$

Therefore, the problem of finding a largest bipartite subgraph  $(V, [A, B]_G)$  is equivalent to the problem of maximizing the measure  $\mu_G$  over all bipartitions of  $V$ .

We are interested in switching location of vertices of a set  $X \subset V$  relatively to the given  $(A, B) \in Bp(G)$ . We will use the symbol  $(A, B) \Delta X$  to denote the pair  $(A \Delta X, B \Delta X)$ , where  $\Delta$  is the symmetric difference of sets. We call  $|X|$  the *size of the switching*. A switching location of a vertex  $v$  with a subset  $W$  of its neighbours in the bipartite subgraph is called  $(v, W)$ -switching. It changes  $(A, B)$  into  $(A, B) \Delta (v, W) := (A, B) \Delta (\{v\} \cup W)$ . For graphs from  $\Gamma$ , each switching  $(v, W)$  has the size no greater than 4. Of course, if  $\mu(v, A, B) < 0$  then  $(v, \emptyset)$ -switching improves  $(A, B)$ . We shall refer to  $b_n$ , for  $n = 0, 1, 2, 3$ , as the *max-cut number guaranteed by the algorithm  $\mathcal{A}_n$*  which uses switchings  $(v, W)$  with  $|W| \leq n$ . In other words,  $b_n$  is the largest number such that for each output  $(A, B)$  of the algorithm  $\mathcal{A}_n$  we have  $|[A, B]_G| \geq b_n|E(G)|$  for every  $G \in \Gamma$  and this lower bound is sharp.

For any graph  $G = (V, E)$ , it has been proved in [3], that the algorithm which uses  $(v, W)$ -switchings with  $|W| \leq 1$  obtains Edwards lower bound i.e.  $b_1|E| \geq \frac{1}{2}|E| + \frac{1}{8}(\sqrt{8|E| + 1} - 1)$ . Additionally, it has been shown, that in the general case the concept of  $(v, W)$ -switchings has its limits no matter how large  $W$  is taken. Remark, that the switching location algorithms can be used as heuristics which also solve the graph bisection problem (see [9]).

In this paper, we investigate local switchings in  $\Gamma$  or its subclass of graphs without pendant triangles. We ask for local switching algorithm which obtains  $(A, B)$  such that

$$|[A, B]_G| \geq \frac{3}{4}|E(G)| - \frac{1}{4}, \quad (2)$$

i.e. the lower bound given in [2] for an arbitrary connected  $G \in \Gamma$ .

The paper is organized as follows. In Section 2, using the function  $\mu$  on vertices we define  $(m, n)$ -algorithms — a subclass of 3-star searching algorithms. The algorithms start with an arbitrary vertex partition and make local changes to improve the current partition, provided that some (local) conditions are satisfied. Each switching realized by the  $(m, n)$ -algorithm has the size no greater than  $m + n + 2$ .

In Section 3 we extend the Bondy–Lock concept of D-paths on the case of graphs with triangles. Some characterizations of the output sets of  $(m, n)$ -algorithms are given in Theorems 2–4. In the main Theorem 4 of the paper, we show that, for graphs without pendant triangles, the  $(2, 2)$ -algorithm obtains the lower bound  $\frac{3}{4}$  of edges. It is the simplest  $(m, n)$ -algorithm for this lower bound because of Theorem 3. Finally, in Section 4, we discuss the limits of this approach, showing that there exists an infinite class of connected graphs from  $\Gamma$ , for which the lower bound (2) cannot be attained no matter how large fixed-size switchings are taken.

## 2. Switching location of vertices and $(m, n)$ -algorithms

Let us denote

$$M(v, W, A, B) := \mu(v, A, B) + \sum_{u \in W} \mu(u, A, B) - 2(|W| - |[W, W]|). \quad (3)$$

It is easy to check, that a  $(v, W)$ -switching improves  $(A, B)$  if and only if  $M(v, W, A, B) < 0$  (we say that it is an *improving switching*). Furthermore, if the equalities take place, then it is *indifferent switching*.

For  $(A, B) \in Bp(G)$  let us define the following families of sets:

$$S_n(A, B) := \{(v, W) \mid |W| = n \text{ and } M(v, W, A, B) < 0\} \quad (4)$$

and

$$S_n^0(A, B) := \{(v, W) \mid |W| = n \text{ and } M(v, W, A, B) = 0\} \quad (5)$$

of improving and indifferent switchings, respectively.

Each  $(v, W)$  induces in  $G$  the structure which consists of: the subgraph of  $G$  induced by the set  $\{v\} \cup W$  and the values of  $\mu$  in each vertex. Structures for the switchings from  $S_n(A, B)$  and from  $S_n^0(A, B)$  are members of the set  $\alpha_n$  in Fig. 1 and of the set  $\beta_n$  in Fig. 2, respectively. Each improving switching increases the measure of effectiveness at least by 4 while each indifferent switching does not change the measure.



We have the following obvious relations between  $\mu$ -algorithms for  $G \in \Gamma$  and  $n, m \in \{1, 2, 3\}$ :

$$A_{m,n}(G) \subset A_{m-1,n}(G) \subset A_n(G) \quad \text{and} \quad A_{m,n}(G) \subset A_{m,n-1}(G).$$

Analogously with  $b_n$ , we shall refer to  $b_{m,n}$  as the max-cut number guaranteed by the  $(m, n)$ -algorithm in the class  $\Gamma$ .

### 3. D-paths and vertices with respect to the strength functions

Let  $(A, B) \in A_0$ . Denote by  $D_i$ , for each  $i = 0, 1, 2, 3$ , the set of vertices

$$D_i := \{v \in V(G) \mid \mu(v, A, B) = i\} \quad \text{and} \quad D_1^* := \{v \in D_1 \mid d(v, V) = 1\}.$$

It is easy to see, that a vertex  $v \in D_0 \cup D_2$  if and only if  $v$  has degree 2 in  $G$ . We will use the maximal (by inclusion) paths with all internal vertices of degree 2 in the bipartite subgraph with  $(A, B)$ . Each such path is an alternating path. We identify the paths which go through the same set of edges. For convenience (see [1,7]), we shall refer to these paths as *D-paths*. Of course, D-paths are edge disjoint. Then we look at D-paths as subnetworks of the bipartite subgraph. The set of all D-paths — subgraphs for a bipartition  $(A, B)$  will be denoted by  $D(A, B)$ . In this convention, each D-path  $P \in D(A, B)$  with ends in  $D_0 \cup D_3 \cup D_1^*$  has exactly two representations as sequences of vertices. D-paths with all vertices in  $(D_1 \setminus D_1^*) \cup D_2$  are cycles which have more representations and they form components of the bipartite subgraph. For D-path  $P$ , we write also  $P = (v_1, \dots, v_n)$  when no confusion can arise.

#### 3.1. Properties of D-paths

It will be convenient, to represent D-paths also by the sequences of the  $\mu$ -weights of its vertices. For  $P = (v_1, \dots, v_n)$  we set  $s(P) = (s_1, \dots, s_n)$  such that  $s_i = \mu(v_i, A, B)$  for each  $i = 1, \dots, n$ . Using these sequences we define the following two functions: the *weight* (more precisely  $\mu$ -weight) of  $P$

$$w(P) := \begin{cases} \sum_{i=1}^{n-1} s_i & \text{if } v_1 = v_n, s_n \neq 3, \\ \sum_{i=1}^n s_i - 2|\{i \mid s_i = 3\}| & \text{otherwise} \end{cases}$$

and the *length* ( $\mu$ -length) of  $P$

$$l(P) := n - 1 + \frac{1}{2} |\{i \leq n \mid s_i \in \{0, 1\} \text{ and } v_i \notin D_1^*\}|.$$

It is easy to check that  $w(P)$  and  $l(P)$  do not depend on the representation of the D-path  $P$  by the sequence of vertices. We shall refer the pair  $(w(P), l(P))$  as the *strength* of  $P$ . Also, it will be useful to look at the strength as a number. We define the *numerical strength* of  $P$  as

$$\tau(P) := w(P) - l(P).$$

The intention of the definitions of the  $\mu$ -weight and the  $\mu$ -length given above was to obtain the following important property of the numerical strength.

**Lemma 1.** *For every  $(A, B) \in \Lambda_0(G)$ , we have*

$$\mu_G(A, B) - |E(G)| = \sum_{P \in D(A, B)} \tau(P).$$

**Proof.** D-paths are edge-disjoint and each edge of the bipartite subgraph is contained in exactly one D-path. Each of the two edges outside  $[A, B]$  are vertex-disjoint because  $(A, B) \in \Lambda_0$ . Additionally,  $v \in D_0 \cup (D_1 \setminus D_1^*)$  if and only if  $v$  is adjacent to an edge from outside of  $[A, B]$ . Therefore,

$$|E(G)| = |[A, B]| + \frac{1}{2}(|D_0| + |D_1 \setminus D_1^*|) = \sum_{P \in D(A, B)} l(P).$$

If  $v$  is a common vertex of the two different D-paths then  $v \in D_3$ . According to the definition of the weight, we have  $\mu_G(A, B) = \sum_{P \in D(A, B)} w(P)$ , and the lemma follows.

On the other hand, the representation of D-path  $s(P)$  contains an information about edges outside the set  $[A, B]_G$ .  $\square$

**Lemma 2.** *Let  $(A, B) \in \Lambda_2(G)$  and  $P = (v_1, \dots, v_n)$  be a D-path such that  $s(P) = (s_1, \dots, s_n)$ . If  $s_{i-1} + s_i + s_{i+1} \leq 3$ , then  $\{v_{i-1}, v_{i+1}\} \in E(G)$ .*

**Proof.** If the statement is not true then, by (3) and (4),  $(v_i, \{v_{i-1}, v_{i+1}\})$  is an improved switching (forms a forbidden structure from  $\alpha_2$  in Fig. 1) for  $(A, B) \in \Lambda_2(G)$ , in spite of the assumption.  $\square$

**Corollary 1.** *Let  $(A, B) \in \Lambda_{1,2}(G)$  and  $P$  be a D-path. Suppose that in  $s(P) = (s_1, \dots, s_n)$  we can find one of the following subsequences  $(\dots, 1, 1, 1, 1, s_i, \dots)$ ,  $(\dots, 1, 1, 1, 1, 2, s_i, \dots)$  or  $(\dots, 1, 1, 1, 1, 2, 2, s_i, \dots)$ , then  $s_i \geq 2$  in each case (we have the same situation if we reverse any of the indicated subsequence). If, additionally,  $G$  has no pendant triangles then for  $(\dots, 0, 2, s_i, \dots)$  we have  $s_i \geq 2$ .*

### 3.2. Worse D-paths for bipartitions from $\Lambda_{1,2}$

According to (1) and Lemma 1, a bipartite subgraph has more edges if its D-paths have larger numerical strengths.

**Theorem 1.** *Let  $G \in \Gamma$  (or, additionally,  $G$  has no pendant triangles) and  $(A, B) \in \Lambda_{1,2}(G)$ . If  $P$  is a D-path and  $\tau(P) < 1$ , then  $P$  is one of the path indicated in Tables 1 and 2 (only in Table 1).*

**Proof.** We will use  $\sigma_{i,j}$  to denote the sequence of  $i \geq 0$  consecutive 1's and  $j \geq 0$  consecutive 2's,  $i + j > 0$ . It will be called a *segment*. For  $P = (p_1, \dots, p_n)$  with

Table 1

All D-paths with the numerical strength less than 1 in graphs without pendant triangles and  $(A, B) \in A_{1,2}$ . Define  $\sigma^{0,k} = (1, 1, 2, \dots, 1, 1, 2)$  with  $k$  times of  $(1, 1, 2)$

Name of $P$	$s(P)$	$(w(P), l(P))$	$\tau(P)$	Remarks
$\pi_1$	$(3, 1, 1, 1, 1, 3)$	$(6, 7)$	$-1$	
$\pi_2$	$(3, 1, 1, 1, 3)$	$(5, 5\frac{1}{2})$	$-\frac{1}{2}$	Also as a cycle
$\pi_3$	$(3, 0)$	$(1, 1\frac{1}{2})$	$-\frac{1}{2}$	
$\pi_4$	$(3, 2, 1, 1, 1, 1, 3)$	$(8, 8)$	$0$	Also as a cycle
$\pi_5^k$	$(3, \sigma^{0,k}, 1, 1, 3)$	$(4 + 4k, 4 + 4k)$	$0$	$k \geq 0$
—	$(3, 1, 1, 1, 1, 2, 2, 2, 1, 1, 1, 1, 3)$	$(16, 16)$	$0$	
$\pi_6$	$(3, 2, 1, 1, 1, 3)$	$(7, 6\frac{1}{2})$	$\frac{1}{2}$	
$\pi_7^k$	$(3, \sigma^{0,k}, 1, 3)$	$(3 + 4k, 2\frac{1}{2} + 4k)$	$\frac{1}{2}$	$k \geq 0$
$\pi_8$	$(3, 2, 0)$	$(3, 2\frac{1}{2})$	$\frac{1}{2}$	
$\pi_9^k$	$(3, \sigma^{0,k}, 1, 1)$	$(3 + 4k, 2\frac{1}{2} + 4k)$	$\frac{1}{2}$	$k \geq 0, v_1 \in D_1^*$
—	$(3, 1, 1, 1, 1, 2, 2, 2, 1, 1, 1, 3)$	$(15, 14\frac{1}{2})$	$\frac{1}{2}$	
$C_1$	$(2, 2, 1, 1, 1, 1, 2)$	$(8, 8)$	$0$	D-cycle
$C_2^k$	$(2, \sigma^{0,k}, 1, 1, 2)$	$(4 + 4k, 4 + 4k)$	$0$	With odd $k$
$C_3^k$	$(2, 1, 2, \sigma^{0,k}, 1, 1, 2)$	$(7 + 4k, 7 + 4k)$	$0$	With odd $k$

Table 2

D-paths with the numerical strength less than 1 in graphs with pendant triangles and  $(A, B) \in A_{1,2}$

Name of $P$	$s(P)$	$(w(P), l(P))$	$\tau(P)$	Remarks
$\pi_{10}^k$	$(0, 2, \sigma^{0,k}, 0)$	$(2 + 4k, 3 + 4k)$	$-1$	$k \geq 1$
$\pi_3^k$	$(3, \sigma^{0,k}, 0)$	$(1 + 4k, 1\frac{1}{2} + 4k)$	$-\frac{1}{2}$	$k \geq 1$
$\pi_{11}^k$	$(3, 1, 2, \sigma^{0,k}, 0)$	$(4 + 4k, 4 + 4k)$	$0$	$k \geq 0$
$\pi_8^k$	$(3, 2, \sigma^{0,k}, 0)$	$(3 + 4k, 2\frac{1}{2} + 4k)$	$\frac{1}{2}$	$k \geq 1$
$\pi_{12}^k$	$(3, 1, 2, 1, 2, \sigma^{0,k}, 0)$	$(7 + 4k, 6\frac{1}{2} + 4k)$	$\frac{1}{2}$	$k \geq 0$

$s(P) = (s_1, \dots, s_n)$ , we define  $s^*(P) := \sigma$  if  $P$  has no ends in  $D_3 \cup D_0 \cup D_1^*$  and, otherwise,  $s^*(P) := (s_1, \sigma, s_n) = s(P)$ , where  $\sigma := (\sigma^1, \dots, \sigma^r), r \geq 1$  is the sequence of segments ( $\sigma = \emptyset$  for  $n = 2$ ). For  $s^*$  to be a one–one mapping of the sequence representations of D-paths into segment presentation assume that  $\sigma$  satisfies the following conditions:

(\*) if  $\sigma^k = \sigma_{0,j}$  or  $\sigma^k = \sigma_{i,0}$  then  $k = 1$  or  $k = r$ , respectively.

For example, if  $s(P) = (0, 2, 2, 1, 1, 2, 1, 1, 3)$  then  $s^*(P) = (0, \sigma_{0,2}, \sigma_{2,1}, \sigma_{2,0}, 3)$ .

By Corollary 1, it is obvious that each  $\sigma_{i,j}$  in  $s^*(P)$  has  $i \leq 4$ . We say that a segment  $\sigma_{i,j}$  is *bordering* if it can be placed only as the first or as the last in  $\sigma$ . It is easy to check that except bordering segmentents by (\*), the following  $\sigma_{4,1}, \sigma_{3,1}, \sigma_{4,2}$  (also  $\sigma_{4,3}$  if  $(A, B) \in A_{2,2}$ ) have to be placed as the right-bordering with  $s_n = 3$ .

It is natural to define the weight and the length of segments in the following way  $w(\sigma_{i,j}) := i + 2j$  and  $l(\sigma_{i,j}) := \frac{3}{2}i + j$ . An easy computation shows that for  $s^*(P) =$

Table 3

All segments with the numerical strengths not greater than 2 in  $s^*(P)$  of D-paths for  $(A, B) \in A_{1,2}$ 

$(s_1, s_n)$ :	$\emptyset$	(3,3)	(3,1)	(1,3)	(3,0)	(0,3)	(1,0)	(0,1)	(0,0)
$\tau((s_1, s_n))$ :	0	1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-2
Bordering $\sigma_{i,j}$ :	$\sigma_{4,0}$	$\sigma_{3,0}$	$\sigma_{2,0}$	$\sigma_{4,1}$	$\sigma_{3,1}$	$\sigma_{1,0}$	$\sigma_{4,2}$	$\sigma_{0,j}$	
$\tau(\sigma_{i,j})$ :	-2	$-1\frac{1}{2}$	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$j$	
Other $\sigma_{i,j}$ :	$\sigma_{2,1}$	$\sigma_{1,1}$	$\sigma_{3,2}$	$\sigma_{2,2}$	$\sigma_{1,2}$	$\sigma_{3,3}$	$\sigma_{2,3}$	$\sigma_{4,4}$	$\sigma_{4,3}$
$\tau(\sigma_{i,j})$ :	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	$1\frac{1}{2}$	2	2	1

$(s_1, \sigma, s_n)$  we have  $\tau(P) = \tau((s_1, s_n)) + \sum_{k=1}^r \tau(\sigma^k)$ , where the numerical strengths are the differences of the weights and the lengths of the segments and the one edge D-path  $(s_1, s_n)$  (also given in Table 3).  $\square$

Here are some elementary properties of these presentation of D-paths.

1. For every sequence  $\sigma$  of nonbordering (interior) segments we have  $\tau(\sigma) \geq 0$  with equality if and only if  $\sigma$  is the sequence with  $k$  times of the same segments of  $\sigma_{2,1}$  (denoted  $\sigma^{0,k}$ ).
2. If  $\sigma_{i,j}$  is the predecessor of a bordering  $\sigma_{3,t}$  (or  $\sigma_{4,t}$ ), then  $j \geq 2$  (or  $j \geq 3$ , respectively).

It is easy to choose all D-paths with negative numerical strengths (see Tables 1 and 2), but many cases should be checked. We skipped it, because of the size restriction of the paper.

**Remark 1.** Each D-path except the two without names in Table 1 can be also found in the case  $(A, B) \in A_{3,3}$ .

### 3.3. The strengths of vertices with respect to the bipartition

Analysing the configuration of D-paths from a vertex  $v \in D_3$  we can deduce about edges outside  $[A, G]_G$ . For example, if  $P = (v, p)$  and  $Q = (v, q)$  are both of  $\pi_3$  type and  $(A, B) \in A_2$ , then  $(p, q) \in E(G)$  (see the last structure in  $\alpha_2$  in Fig. 1).

**Lemma 3.** Let  $(A, B) \in A_{1,2}$ ,  $v \in D_3$  and  $P = (v, p_1, \dots, p_{n_1})$ ,  $Q = (v, q_1, \dots, q_{n_2})$  and  $U = (v, u_1, \dots, u_{n_3})$ . We have

1. If  $s(P) = (3, 1, 1, \dots)$  and  $s(Q) = (3, 1, 1, \dots)$ , then  $\{p_1, q_1\} \in E(G)$  or  $\{p_2, q_2\} \in E(G)$ .
2. If  $s(P) = (3, 1, 1, \dots)$ ,  $s(Q) = (3, 1, \dots)$  and  $s(U) = (3, 1, \dots)$ , then the subgraph induced by the set of vertices  $\{p_1, q_1, u_1\}$  has an edge.

**Proof.** If 1 or 2 are not true, then  $(A, B) \Delta \{p_1, p_2\} \notin A_2$  and so on  $(A, B) \notin A_{1,2}$  in spite of the assumption.



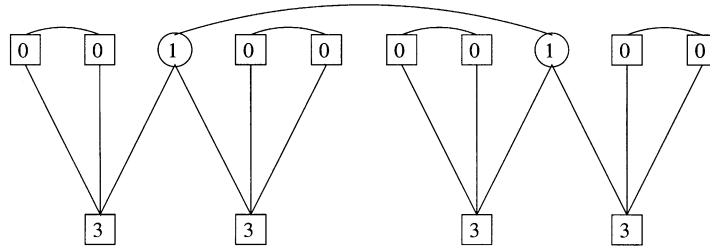


Fig. 3. Graph  $G$  and  $(A, B) \in A_{1,3}$ . We represent a vertex  $v \in (D_1 \setminus D_1^*) \cup D_2$  by a circle and  $v \in D_0 \cup D_3 \cup D_1^*$  by a box including its value  $\mu(v, A, B)$ .

For  $v \in D_3 \cup D_0 \cup D_1^*$ , let  $Dp(v)$  denotes the sequence of D-paths from the vertex  $v$ . Of course,  $Dp(v)$  has exactly one element for each  $v \in D_0 \cup D_1^*$ . In the case  $v \in D_3$ , it will be convenient to denote  $Dp(v) = (P, Q, U)$  with three elements even if  $v$  belongs to a D-path which is a cycle. Also, we write  $Dp(v)$  as sequences of types of its D-paths.

The *strength of the vertex*  $v \in D_3 \cup D_0 \cup D_1^*$  is defined as the sum of strengths of paths from  $Dp(v)$ , i.e.  $\omega(v) := (\omega_1(v), \omega_2(v))$ , where the first and second coordinates are the sum of weights and lengths of D-paths from  $v$ , respectively. As an example, for  $Dp(v) = (P, Q, U)$  we have  $\omega(v) = (w(P) + w(Q) + w(U), l(P) + l(Q) + l(U))$ .  $\square$

### 3.4. Locale worse-case analysis of $A_{1,2}$

For  $v \in D_3$  it will be convenient to define the *extended strength*  $\omega^+(v)$  as the sum of the strength of  $v$  and the strengths of all  $u \in D_0 \cup D_1^*$  which are the ends of the paths from  $Dp(v)$ . Let us partite the set of all D-paths  $D(A, B) = D'(A, B) \cup D''(A, B)$ , so that  $D'(A, B)$  (or simpler  $D'$ ) is the set of all D-paths which have an end in  $D_3$ .

For example, for  $G$  and  $(A, B)$  in Fig. 3, we have  $D_2 = D_1^* = \emptyset$ ,  $|D_3| = 4$ ,  $|D_0| = 8$  and  $D''(A, B) = \emptyset$ . For each vertex  $v \in D_3$  (or  $u \in D_0$ ) we have  $Dp(v) = (\pi_3, \pi_3, \pi_7^0)$  (or  $Dp(u) = (\pi_3)$ ) and, respectively,  $\omega(v) = (5, 5\frac{1}{2})$  (or  $\omega(u) = (1, 1\frac{1}{2})$ ). The extended strength of each  $v \in D_3$  is equal  $\omega^+(v) = \omega(v) + 2\omega(u) = (7, 8\frac{1}{2})$ .

We rewrite Lemma 1 in the following way:

**Lemma 4.** For every  $(A, B) \in A_0(G)$  we have

$$\frac{\mu_G(A, B)}{|E(G)|} = \frac{\frac{1}{2} \sum_{v \in D_3} \omega_1^+(v) + \sum_{P \in D''} w(P)}{\frac{1}{2} \sum_{v \in D_3} \omega_2^+(v) + \sum_{P \in D''} l(P)}.$$

**Proof.** In the sums over vertices from  $D_3$  each D-path from  $D'(A, B)$  has been counted twice. Then

$$\left( \sum_{v \in D_3} \omega_1^+(v), \sum_{v \in D_3} \omega_2^+(v) \right) = 2 \left( \sum_{P \in D'} w(P), \sum_{P \in D'} l(P) \right)$$

and the equality follows from Lemma 1.

The numerical strength of a subset of vertices  $X \subset D_3$  we define as

$$\tilde{\tau}(X) := \frac{\sum_{v \in X} \omega_1^+(v)}{\sum_{v \in X} \omega_2^+(v)} \quad \text{and} \quad \tilde{\tau}(v) := \tilde{\tau}(\{v\}). \quad \square$$

**Corollary 2.** Let  $(A, B) \in A_0(G)$ . Let  $r$  be a real number such that  $w(P)/l(P) \geq r$  for every  $P \in D''$ . If  $D_3 = X_1 \cup \dots \cup X_k$ ,  $k \geq 1$ , is a partition (i.e.  $X_i \cap X_j = \emptyset$  for each  $i \neq j$ ) and for each  $i = 1, \dots, k$  we have  $\tilde{\tau}(X_i) \geq r$ , then  $\mu_G(A, B) \geq r|E(G)|$ .

In the worse-case analysis we look for vertices  $v \in D_3$  with the numerical strengths  $\tilde{\tau}(v) < 1$  or, in other words, with  $\omega_1^+(v) < \omega_2^+(v)$ . According to the example in Fig. 3, we have  $\tilde{\tau}(v) = \frac{14}{17}$  for each  $v \in D_3$ .

**Lemma 5.** Let  $G$  be without pendant triangles and  $(A, B) \in A_{1,2}(G)$ . If  $v \in D_3$  then either  $\omega_1(v) \geq \omega_2(v)$  or one of the following statements is true:

- $Dp(v) = (\pi_1, \pi_4, U)$  with  $U$  of the type  $\pi_4$ ,  $\pi_6$  or  $\pi_7^k$ ,  $k = 0, 1, \dots$ ,
- $Dp(v) = (\pi_2, \pi_4, U)$  with  $U$  of the type  $\pi_4$  or  $\pi_5^k$ ,  $k = 0, 1, \dots$ ,
- two  $D$ -paths of the type  $\pi_2$  along with  $\pi_4$  or  $\pi_6$  (also  $\pi_8$  if only the two forms a cycle) start from  $v$ .

**Proof.** We will denote, as above,  $Dp(v) = (P, Q, U)$ ,  $P = (v, p_1, \dots, p_{n_1})$ ,  $Q = (v, q_1, \dots, q_{n_2})$  and  $U = (v, u_1, \dots, u_{n_3})$ . Also, we assume for numerical strengths  $\tau(P) \leq \tau(Q) \leq \tau(U)$ . Suppose there exists  $v \in D_3$  such that  $\omega_1(v) < \omega_2(v)$ . Theorem 1 now leads to  $P \in \{\pi_1, \pi_2, \pi_3\}$ .

*Point 1:*  $P = \pi_1$ . From Lemma 3.1, both  $s(Q)$  and  $s(U)$  are different from  $(3, 1, 1, \dots)$  and  $(3, 0)$ . Therefore,  $Q = \pi_4$  because of Theorem 1. In the last case  $U \in \{\pi_4, \pi_6, \pi_7^k\}$ ,  $k = 0, 1, \dots$ .

*Point 2:*  $P = \pi_3$ . If  $Q = \pi_3$  then  $\{p_1, q_1\} \in E(G)$  and  $v$  is in a pendant triangle, in spite of the assumption. If  $s(Q) = (3, 1, 1, s_4, \dots)$  or  $s(Q) = (3, 2, 1, 1, s_5, \dots)$ , we would have  $\{p_1, q_1\} \in E(G)$  with  $s_4 \geq 2$  or  $\{p_1, q_3\} \in E(G)$  with  $s_5 \geq 2$ , respectively. Additionally, in both cases, if  $s(U) = (3, 1, s_3, \dots)$  then  $s_3 \geq 2$ . We conclude from Theorem 1 that  $\omega_1(v) \geq \omega_2(v)$ .

*Point 3:*  $P = \pi_2$ . From Lemma 3.1 we conclude that it is not possible that  $s(Q)$  together with  $s(U)$  be of the type  $(3, 1, 1, \dots)$ . If there is not more than one  $\pi_2$  in  $Dp(v)$  then  $Dp(v) = \{\pi_2, \pi_4, \pi_4\}$  or  $Dp(v) = \{\pi_2, \pi_4, \pi_5^k\}$ . Let  $Q = \pi_2$ . If  $s(U) = (3, i, \dots)$  then  $i \geq 2$  and  $U$  can be a member of  $\{\pi_4, \pi_6\}$ . If  $U = \pi_8$  then  $\{p_2, u_2\} \in E(G)$  and  $\{q_2, u_2\} \in E(G)$ . It is possible only in the case where  $P$  and  $Q$  represent the same  $D$ -cycle. This ends the proof of the Lemma.  $\square$

**Remark 2.** We have  $\omega^+(v) = \omega(v)$  for each vertex indicated in Lemma 5 except of the one with  $Dp(v) = (\pi_2, \pi_2, \pi_8)$ . In the last case  $\tilde{\tau}(v) = 1$  and the  $D$ -paths from  $Dp(v)$  form a component, say  $G^*$ , of  $G$ .

Therefore, we can add to Lemma 5 the following statement.

**Lemma 6.** *In the case without pendant triangles, for each  $(A, B) \in A_{1,2}$  and  $v \in D_3$  such that  $\omega(v) \neq \omega^+(v)$ , we have either  $\tilde{\tau}(v) \geq 1$  or  $Dp(v) = (\pi_3, \pi_5^k, \pi_7^l)$  where  $k, l = 0, 1, \dots$ .*

**Proof.** A new configuration of D-paths can be obtained in the case  $\pi_3 \in Dp(v)$  and  $\omega_1(v) = \omega_2(v)$ . We can find them by the same arguments as in Point 2 above.  $\square$

**Lemma 7.** *Let  $G \in \Gamma$  and  $(A, B) \in A_{1,2}(G)$ . If  $v \in D_3$  and there is a D-path in  $Dp(v)$  which goes through a pendant triangle, then either  $\tilde{\tau}(v) \geq 1$  or one of the following:*

- $Dp(v) = (\pi_1, \pi_{11}^k, U)$  with  $k \geq 0$  and  $U$  of the type  $\pi_4$  or  $\pi_6$ ;
- the cycle  $\pi_2$  along with  $\pi_{11}^k$  starts from  $v$  or  $Dp(v) = (\pi_2, \pi_{11}^k, \pi_4)$ ,  $k \geq 0$ ;
- $Dp(v) = (\pi_3^k, \pi_i^l, \pi_j^m)$ , where  $k, l, m \geq 0$  ( $\pi_3^0 := \pi_3$ ),  $i, j \in \{5, 7, 11\}$  and  $i \neq j$ ;
- $Dp(v) = (\pi_3^k, \pi_3^l, U)$  where  $k, l \geq 0$ ,  $\tau(U) < 2$  and  $s(U)$  different from  $(3, 1, 1, \dots)$  and  $(3, 2, 1, 1, \dots)$ .

**Proof.** By Theorem 1, we look for vertices  $v \in D_3$  such that D-paths indicated in Tables 1 and 2 belong to  $Dp(v)$  and  $\omega_1^+(v) < \omega_2^+(v)$ . We use the same arguments as in the proof of Lemma 5.

#### 4. Valuation of $(m, n)$ -algorithms

In the proofs of the theorems given below we analyze all the worse cases indicated in Lemmas 5–7.

**Remark 3.** For every three nonnegative real numbers  $a \leq b$  and  $x$  we have  $(a + x/b + x) \geq a/b$ . Therefore, in the case  $\tilde{\tau}(v) < 1$  and  $\pi_i^k \in Dp(v)$  for some  $k > 0$  we have: if  $\tilde{\tau}(v)$  can be written as  $(a_1 + w(\pi_i^k))/(a_2 + l(\pi_i^k))$ , then

$$\tilde{\tau}(v) \geq \frac{a_1 + w(\pi_i^{k-1})}{a_2 + l(\pi_i^{k-1})}.$$

Particularly, from the set of possible  $\{\pi_i^k\}$  in the worse case analysis we can choose the one with the minimal  $k$ .

**Theorem 2.** *For the class  $\Gamma$ , the max-cut number guaranteed by (1,2)-algorithm or (1,3)-algorithm are  $b_{1,2} = b_{1,3} = \frac{12}{17}$ .*

**Proof.** Let  $(A, B) \in A_{1,2}(G)$ . For inequality  $|[A, B]_G| \geq \frac{12}{17}|E(G)|$  it will be enough to show that  $\mu_G(A, B) \geq \frac{14}{17}|E(G)|$ , because of (1).  $\square$

Assume, without loss of generality, that  $G$  is connected. Suppose there is a D-path of the negative numerical strength and without vertices in  $D_3$ . From Theorem 1, there are  $\pi_{10}^k$ ,  $k \geq 1$ . Each  $\pi_{10}^k$  forms a component of  $G$  with  $\omega(\pi_{10}^k) = (2 + 4k, 3 + 4k)$ . We have  $(\mu_G(A, B)/|E(G)|) \geq \frac{6}{7}$ , because of Remark 3. Let  $v \in D_3$ . If  $\omega_1^+(v) < \omega_2^+(v)$  then  $Dp(v)$  is one of the types indicated in Remark 2 and Lemmas 5–7. Analyzing all the worse cases, we verify (see also Remark 3), that in the set of vertices with  $\omega_1^+(v) - \omega_2^+(v)$  equal to  $-2$ ,  $-1\frac{1}{2}$ ,  $-1$  and  $-\frac{1}{2}$ , the minimum of  $\tilde{\tau}(v)$  is obtained for the vertex with  $Dp(v)$  equal to  $(\pi_3, \pi_3, \pi_{11}^0)$ ,  $(\pi_3, \pi_3, \pi_7^0)$ ,  $(\pi_3, \pi_3, (3, 3))$  and  $(\pi_3, \pi_3^0, \pi_7^0)$  (or  $(\pi_3, \pi_3, (3, 1, 2, 3))$ , respectively.

We have  $\tilde{\tau}(v) \geq \frac{14}{17}$  in each case, with equality only in the case  $Dp(v) = \{\pi_3, \pi_3, \pi_7^0\}$ . Therefore,  $\tilde{\tau}(v) \geq \frac{14}{17}$  for each  $v \in D_3$  and, from Corollary 2,  $b_{1,2} \geq \frac{12}{17}$ .

We have  $b_{1,3} \leq \frac{12}{17}$  because of the graph in Fig. 3 with  $|[A, B]| = \frac{12}{17}|E(G)|$ , which completes the proof.  $\square$

#### 4.1. $(m, n)$ -algorithms for the graphs without pendant triangles

The effectiveness of (1,2) and (1,3) algorithms is established by the following theorem.

**Theorem 3.** *For the graphs from  $\Gamma$  and without pendant triangles the max-cut number guaranteed by (1,2)-algorithm or (1,3)-algorithm are  $b_{1,2} = b_{1,3} = \frac{14}{19}$ .*

**Proof.** Let  $G = (V, E)$  has no pendant triangles and  $(A, B) \in A_{1,2}(G)$ . For the inequality  $|[A, B]| \geq \frac{14}{19}|E|$  it will be enough to show  $(\omega_1^+(v)/\omega_2^+(v)) \geq \frac{18}{19}$  for each  $v \in D_3$ , because of Corollary 2 and Eq. (1).

From Theorem 1,  $(w(P)/l(P)) \geq 1$  for every D-path  $P$  which has both ends outside  $D_3$ .

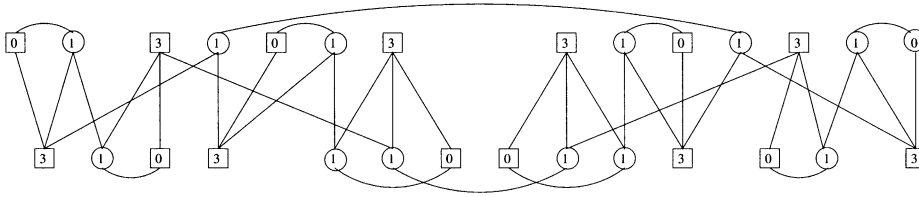
Let  $v \in D_3$ . If  $\omega_1^+(v) < \omega_2^+(v)$  then  $Dp(v)$  is one of the types indicated in Lemmas 5–6 (see also Remark 2). Analysing all the worse cases, we verify (see also Remark 3), that for the set of vertices with  $\omega_1^+(v) - \omega_2^+(v)$  equal to  $-1$  and  $-\frac{1}{2}$  the minimum of  $\tilde{\tau}(v)$  is obtained for the vertex with  $Dp(v)$  equal to  $(\pi_2, \pi_2, \pi_4)$  and  $(\pi_3, \pi_5^0, \pi_7^0)$ , respectively.

Hence  $\tilde{\tau}(v) \geq \frac{18}{19}$  in each case, with equality only for the vertices indicated above. Corollary 2 implies the lower bound.

We have  $b_{1,3} \geq b_{1,2} \geq \frac{14}{19}$ . On the other hand,  $|[A, B]| = \frac{14}{19}|E(G)|$  because of the graph in Fig. 4, which completes the proof.  $\square$

**Theorem 4.** *For the graphs from  $\Gamma$  and without pendant triangles, the max-cut number guaranteed by (2,2)-algorithm or (3,3)-algorithm are  $b_{2,2} = b_{3,3} = \frac{3}{4}$ .*

**Proof.** Let  $G$  has no pendant triangles and  $(A, B) \in A_{2,2}$ . It is sufficient to prove that  $|[A, B]| \geq |E(G)|$ . Let us partite the set  $D_3(A, B) = D_3^+ \cup D_3^- \cup D_3^0$  where  $D_3^+ = \{v \in D_3 \mid \tilde{\tau}(v) > 1\}$ ,  $D_3^- = \{v \in D_3 \mid \tilde{\tau}(v) < 1\}$  and  $D_3^0 = D_3(A, B) \setminus (D_3^+ \cup D_3^-)$ .

Fig. 4.  $G$  and  $(A, B) \in A_{1,3}$  with  $|[A, B]| = \frac{14}{19}|E(G)|$ .

It is easy to verify that among the worse cases indicated in Lemmas 5 and 6 for  $(A, B) \in A_{2,2}$  only  $Dp(v) = (\pi_3, \pi_5^k, \pi_7^l)$  is possible, i.e.

$$D_3^- = \{v \in D_3 \mid Dp(v) = (\pi_3, \pi_5^k, \pi_7^l), k, l = 0, 1, \dots\},$$

where  $\omega^+(v) = (9 + 4(k + l), 9\frac{1}{2} + 4(k + l))$ .

Consider  $v \in D_3^-$  and denote  $Dp(v) = (P^0, Q^1, P^1) = (\pi_5^{k_0}, \pi_3, \pi_7^{k'_0})$ . If  $v_1$  is the end of  $P^1$ , then either  $v_1 \in D_3^+$  or  $\pi_3 \in Dp(v_1)$  and its end of the weight 0 is adjacent to the last but one vertex of  $P^1$ . Therefore,  $v_1$  is not the end of  $P^0$  and  $Dp(v_1) = (P^1, Q^2, P^2) = (\pi_7^{k_1}, \pi_3, \pi_7^{k'_1})$ ,  $k_1 = k'_0$ . If  $v_2$  is the end of  $P^2$ , then either  $v_2 \in D_3^+$  or  $\pi_3 \in Dp(v_2)$  and its end of the weight 0 is adjacent to the last but one vertex of  $P^2$ . Therefore,  $v_2$  is not the end of  $P^0$ ,  $Dp(v_2) = (P^2, Q^3, P^3) = (\pi_7^{k_2}, \pi_3, \pi_7^{k'_2})$ , and so on. The sequence  $(v_1, v_2, \dots)$  is finite, with some  $v_n \in D_3^+$ , because  $G$  is finite. We say  $v_n$  supports  $v$ .

Therefore, for each  $v \in D_3^-$  there exists exactly one  $u \in D_3^+$  which supports  $v$ . If a vertex  $u$  supports two or three different vertices, then  $Dp(u) = (\pi_7^k, \pi_7^{k'}, U)$  with  $U \neq \pi_3$  or  $Dp(u) = (\pi_7^k, \pi_7^{k'}, \pi_7^{k''})$ , respectively. In the first case we have  $s(U) \neq (3, 1, 1, \dots)$  and by Theorem 1,  $\omega_1(u) - \omega_2(u) \geq 1$ . In the second one,  $\omega(u) = (9 + 4n, 7\frac{1}{2} + 4n)$  for some  $n \geq 0$ .

For each  $u \in D_3$  and  $X(u) = \{u\} \cup \{v \in D_3^- \mid u \text{ supports } v\}$  we have

$$\sum_{v \in X(u)} (\omega_1^+(v) - \omega_2^+(v)) \geq 0.$$

It makes it possible to partite the set  $D_3$  as in Corollary 2 to obtain  $\mu_G(A, B) \geq |E(G)|$  and, therefore,  $b_{2,2} \geq \frac{3}{4}$ . We have  $b_{3,3} \leq \frac{3}{4}$ , see the graph  $G^*$  in Remark 2, which completes the proof.  $\square$

Theorems 3 and 4 indicate  $(2, 2)$ -algorithm as the best  $(m, n)$ -algorithm for the graphs without pendant triangles.

#### 4.2. The limits of the switching algorithms

We know (see [2]) that for each connected graph  $G \in \Gamma$  we have the lower bound (2). The extremal graphs consist of  $n$  disjoint triangles with minimal number of edges

for connectivity. It is easy to show that (3,3)-algorithm does not obtain this lower bound even, if we remove some extremal graphs.

Now, we present a construction, showing that the local switching techniques are not strong enough, in general, to attain the lower bound (2).

**Theorem 5.** *For every natural number  $k$  there exists a connected graph  $G$  with vertex bipartition  $(A, B)$  such that  $|[A, B]_G| < \frac{3}{4}|E(G)| - \frac{1}{4}$  and there exists no switching off at most  $k$  vertices that increases the number of edges generated by the vertex partition.*

**Proof.** We construct an infinite family  $\{(\Phi_n; \tilde{A}_n, \tilde{B}_n)\}_{n=0}^\infty$  of connected graphs  $\Phi_n$ , with suitable defined vertex partitions  $(\tilde{A}_n, \tilde{B}_n)$ . The bipartite subgraph generated by  $(\tilde{A}_n, \tilde{B}_n)$  has less edges than the lower bound. Additionally, there exists no switching off at most  $2n+1$  vertices that increases the number of edges generated by  $(\tilde{A}_n, \tilde{B}_n)$ .

Let us start with the construction of the family  $\{(F_n; A_n, B_n)\}_{n=0}^\infty$ , successively for  $n=0, 1, \dots$ . Each graph  $F_n = (V_n, E_n)$ ,  $n > 0$ , has exactly one vertex, say  $x_n$ , of degree 1. Let  $F_0 = K_1$ ,  $A_0 = \{x_0\}$  and  $B_0 = \emptyset$ .

Assume  $(F_n; A_n, B_n)$  to be constructed. To construct  $(F_{n+1}; A_{n+1}, B_{n+1})$  take three disjoint graphs: two copies of  $F_n$ , say  $(F'_n; A'_n, B'_n)$ ,  $(F''_n; A''_n, B''_n)$  (with vertices of degree 1  $x'_n, x''_n$ , respectively) and  $K_2$ , say  $\{u_{n+1}, x_{n+1}\}$ . Define  $F_{n+1}$  as the graph with  $V_{n+1} = V'_n \cup V''_n \cup \{u_{n+1}, x_{n+1}\}$  and

$$E_{n+1} = E'_n \cup E''_n \cup \{\{x'_n, x''_n\}, \{u_{n+1}, x'_n\}, \{u_{n+1}, x''_n\}, \{u_{n+1}, x_{n+1}\}\}.$$

As the bipartition we take  $A_{n+1} = A'_n \cup A''_n \cup \{x_{n+1}\}$  and  $B_{n+1} = B'_n \cup B''_n \cup \{u_{n+1}\}$ . The graph  $F_{n+1}$  has  $x_{n+1}$  as only one vertex of the degree 1.

It is easy to check that for every  $n=0, 1, \dots$  the graph  $F_n$  has  $3 \times 2^n - 1$  vertices,  $|E_n| = 3 \times 2^n - 4$  edges and  $|[A_n, B_n]| = 3 \times 2^n - 3$  edges in the bipartite subgraph defined by  $(A_n, B_n)$ .

To construct  $(\Phi_n; \tilde{A}_n, \tilde{B}_n)$ ,  $n \geq 1$  take two disjoint copies of  $F_n$ , say  $(F'_n; A'_n, B'_n)$ ,  $(F''_n; A''_n, B''_n)$  (with pendant edges  $\{u'_n, x'_n\}$ ,  $\{u''_n, x''_n\}$ , respectively). Define  $\Phi_n = (\tilde{V}_n, \tilde{E}_n)$  as the graph with the vertex set  $\tilde{V}_n = (V'_n \cup V''_n) \setminus \{x'_n, x''_n\}$  and the edge set

$$\tilde{E}_n = ((E'_n \cup E''_n) \setminus \{\{u'_n, x'_n\}, \{u''_n, x''_n\}\}) \cup \{\{u'_n, u''_n\}\}.$$

As the bipartition, we take  $\tilde{A}_n = (A'_n \cup A''_n) \setminus \{x'_n, x''_n\}$  and  $\tilde{B}_n = B'_n \cup B''_n$ . The graph  $\Phi_n$  has  $|\tilde{E}_n| = 8 \times 2^n - 9$  edges and  $|[\tilde{A}_n, \tilde{B}_n]| = 6 \times 2^n - 8$  edges in the cut  $(\tilde{A}_n, \tilde{B}_n)$ . Therefore, we have

$$|[\tilde{A}_n, \tilde{B}_n]| < \frac{3}{4}|\tilde{E}| - \frac{1}{4}.$$

Finally, we claim that each improving switching of the bipartition  $(\tilde{A}_n, \tilde{B}_n)$  contains an alternating path from a vertex of degree 2 in a pendant triangle to one of the vertices  $u'_n$  or  $u''_n$ . The minimal path has  $2n+1$  vertices, and this completes the proof.  $\square$

## 5. Uncited reference

[5]

## References

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